B.Sc. 5th Semester (Honours) Examination, 2019-20

MATHEMATICS

Course ID : 52117 **Course Code : SHMTH-504-DSE-2**

Course Title : Probability and Statistics

Time: 2 Hours **Full Marks: 40**

The figures in the right hand side margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meaning.

1. Answer *any five* of the following: 2×5=10

(a) If A and B are any two events, then prove that the probability of occurring exactly one of them is given by $P(A) + P(B) - 2P(A \cap B)$.

(b) If the random variable X has the probability density $f(x) = \begin{cases} ke^{-3x} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$

find k and $P(0{\cdot}5 \leq X \leq 1)$.

(c) Find the expectation of the random variable X where $X = \begin{cases} 1 & \text{if } A \text{ happens} \\ 0 & \text{if } A \text{ does not happen.} \end{cases}$

(d) If the random variable X and Y have the same standard deviation, show that $U = X + Y$ and $V = X - Y$ are uncorrelated.

(e) Show that the correlation coefficient ρ satisfies the inequality $-1 \leq \rho \leq 1$.

(f) How that the sample mean is unbiased estimator of population mean.

(g) Show that the distribution function $F(x)$ is monotonic non-decreasing on R.

(h) Define sample characteristic. Write expression for sample variance and sample k-th central moment.

2. Answer *any four* questions from the following: 5×4=20

(a) If the joint probability density of X and Y is given by

$$f(x,y) = \begin{cases} 2 & \text{for } x > 0, y > 0, x + y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

find (i) $P\left(X \leq \frac{1}{2}, Y \leq \frac{1}{2}\right)$

(ii) $P\left(X + Y > \frac{2}{3}\right)$

(iii) $P(X > 2Y)$

(b) State and prove Chebyshev's inequality for a continuous random variable.

(c) If X and Y are independent Poisson variates, show that the conditional distribution of X given $X + Y$ is binomial.

(d) Define normal distribution and find mean and variance of a normal random variate.

(e) If $\overline{X}$ be the sample mean of a random sample $(X_1, X_2, \cdots, X_n)$ drawn from an infinite population with mean μ and variance σ^2 then show that

(i) $E(\overline{X}) = \mu$

(ii) $\text{var}\,(\overline{X}) = \sigma^2/n$, and

(iii) $E\left(\frac{n}{n-1}\, S^2\right) = \sigma^2$ where

S^2 is the sample central moment of order 2.

(f) Obtain the maximum likelihood estimate of θ on the basis of a random sample of size n drawn from the population whose p.d.f. is given by $f(x) = c\, e^{-x/\theta}, 0 \le x < \infty$, where c is a constant and $\theta > 0$. Discuss consistency and un-biasedness of the estimate.

3. Answer *any one* question from the following: 10×1=10

(a) (i) Find the mean and variance of a Poisson random variable with parameter m.

(ii) State central limit theorem for independent and identically distributed random variables with finite variance.

(iii) Consider a Markov chain with state space $\{0, 1, 2, 3\}$ and transition probability matrix
$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
Determine which states are transient and which are recurrent. 5+2+3=10

(b) (i) The marks obtained by 18 candidates in an examination having a mean 56 and variance 65. Find 95% confidence interval for the mean of the population of marks, assuming it to be normal.

[For 17 degrees of freedom $P(|t| > 2{\cdot}11) = 0{\cdot}05$]

(ii) Let p be the probability that a coin will fall head in a single toss in order to test $H_0 : p = \frac{1}{2}$ against $H_1 : p = \frac{3}{4}$. The coin is tossed 5 times and H_0 is rejected if more than 3 heads are obtained. Find the probability of type I error and power of the test. 5+5=10

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MATHEMATICS

Course ID : 52117 **Course Code : SHMTH-504-DSE-2**

Course Title : Boolean Algebra and Automata

Time: 2 Hours **Full Marks: 40**

The figures in the right hand side margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meaning.

1. Answer *any five* of the following: 2×5=10

(a) State the Pumping Lemma for regular languages.

(b) Briefly prove or disprove: Every finite lattice is complete.

(c) Briefly prove or disprove: Homomorphic image of a distributive lattice is distributive.

(d) Define a Turing machine.

(e) Is there a unique DFA corresponding to any state diagram realizing it? If yes, prove it; if no, give example.

(f) An *antichain* is a subset of a partially ordered set such that any two distinct elements in the subset are incomparable.

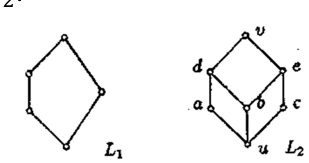
Given an example of an ordered set P such that $|P| > 3$ in which there are three elements x, y, z such that

i) $\{x, y, z\}$ is an antichain.

ii) $x \vee y$, $y \vee z$ and $z \vee x$ fail to exist.

iii) $\vee \{x, y, z\}$ exists.

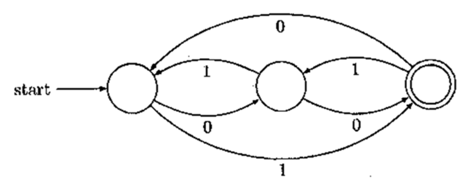
(g) Find L_1 as a sublattice of L_2.



(h) Let $\sum = \{1\}$. Prove that there is an undecidable subset of $\sum^*$.

2. Answer *any four*: 5×4=20

(a) Convert the following DFA to a regular expression.



(b) Let $\sum = \{a, b\}$. Let L be a language over $\sum$, consisting of strings of length at least 2, where the first letter is the same as the last letter, and the second letter is the same as the second to last letter. For example, $a \notin L$, $b \notin L$, $aa \in L$, $aaa \in L$, $aba \in L$, $bbaabba \notin L$. Design a DFA that accepts L.

(c) Draw the state diagram of a Pushdown Automata realizing $\{0^n 1^n | n \geq 0\}$.

(d) Define Kleene closure of a regular language A, as $A^* = \{x_1, x_2 \dots x_k | k \geq 0$ and each $x_i \in A\}$. Prove that Kleene closure of every regular language is regular.

(e) Let P and Q be finite ordered sets and let $\psi : P \to Q$ be a bijective map. Then prove the following to be equivalent:

(i) $x < y$ in P if and only if $\psi\,(x) < \psi\,(y)$ in Q.

(ii) $x \sim < y$ in P if and only if $\psi\,(x) \sim < \psi\,(y)$ in Q.

(f) Prove that any distributive lattice is modular. Is the converse true? Justify. 3+2=5

3. Answer *any one*: 10×1=10

(a) (i) Define, $A_{DFA} = \{< B, \omega > | B$ is a DFA that accepts input string $\omega\}$. Prove that, A_{DFA} is a decidable language.

(ii) Prove that, every nondeterministic Turing machine has an equivalent deterministic Turing machine. 5+5=10

(b) (i) Let $f : B \to C$ where B and C are Boolean algebras.

A. Assume f to be a lattice homomorphism. Then prove the following to be equivalent:

$f(0) = 0$ and $f(1) = 1$.

$f(a') = \big(f(a)\big)'$, $\forall a \in B$.

B. Also prove that, if f preserves $'$, then f preserves $\vee$ if and only if f preserves $\wedge$.

(ii) Let L and K be lattices and $f : L \to K$ a map. Prove the following to be equivalent:

A. f is order preserving.

B. $(\forall a, b \in L)\ f(a \vee b) \geq f(a) \vee\ f(b)$.

C. $(\forall a, b \in L)\ f(a \vee b) \leq f(a) \wedge\ f(b)$. 5+5=10

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