B.SC. FIFTH SEMESTER (HONS.) EXAMINATION 2021

Subject: MATHEMATICS **Course ID: 52112**

Course Title: Group Theory – II **Course Code: SH/MTH/502/C-12**

Full Marks: 40 **Time: 2 Hours**

The figures in the margin indicate full marks

Notations and symbols have their usual meaning

1. **Answer *any FIVE* of the following questions:** $(2 \times 5 = 10)$

a) Find the kernel of a group action.

b) Show that a cyclic group of order 4 cannot be expressed as an internal direct product of two subgroups of order 2.

c) Find the group $Aut(\mathbb{Z}_8)$.

d) Show by an example that the external direct product of two cyclic groups may not be cyclic.

e) Let G be a group and $a \in G$. Then prove that $Z(G) \subseteq C(a)$.

f) Define ap-group with example.

g) Let Q_8 denote the quaternion group of order 8. Find $Z(Q_8)$.

h) Let G be a non-commutative group of order p^3 where p is a prime. Then prove that $|Z(G)| = p$.

2. **Answer *any FOUR* of the following questions:** $(5 \times 4 = 20)$

a) Prove that every group of order $3^2 \cdot 5 \cdot 7$ is abelian.

b) (i) Show that $|\operatorname{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2)| = 6$.

(ii) Prove that the characteristic subgroups are normal. (3+2)

c) (i) Let G be a finite p-group with $|G| > 1$. Then prove that $|Z(G)| > 1$.

(ii) Let G be a finite group of order p^n where p is a prime and $n \geq 1$. Prove that any subgroup of G of order p^{n-1} is a normal subgroup of G. (3+2)

d) (i) Prove that S_3 is not isomorphic to a direct product of two non-trivial groups.

(ii) Prove that $(\mathbb{Q}, +)$ is not isomorphic to a direct product of two non-trivial groups. (3+2)

e) Determine which of the following cannot be the class equation of a group:

(i) 10 = 1+1+1+2+5 (ii) 4 = 1+1+2 (iii) 8 = 1+1+3+3 (iv) 6 = 1+2+3.

f) (i) Show that a Sylow 11-subgroup of G of order 44 is normal in G.

(ii) How many elements of order 7 are there in a simple group of order 168? (2+3)

3. **Answer *any ONE* of the following questions:** $\mathbf{(10 \times 1 = 10)}$

a) (i) Let G be a finite group and let N be a normal abelian subgroup of G. Suppose that the orders of ${}^{G}/_{N}$ and $\mathrm{Aut}(N)$ are relatively prime. Prove that N is contained in the center of G.

(ii) Show that a group of order 96 has a normal subgroup of order 16 or 32.

(iii) Find the order of each element in the group $D_4 \times \mathbb{Z}_2$. (5+3+2)

b) (i) Let G be a group and S be a G-set. Then show that $[G: G_a] = |[a]|\ \ \forall\, a \in S$.

(ii) If G is a simple group of order 60, then show that $G \cong A_5$. (3+7)

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