M.SC. FOURTH SEMESTER EXAMINATIONS, 2021

Subject: Mathematics **Course ID: 42152**

Course Code: Math-402

Course Title: Graph Theory & Field Theory (Old)

Full Marks: 40 **Time: 2 Hours**

The figures in the margin indicate full marks

Notations and symbols have their usual meaning.

Group A – Graph Theory

Answer any three questions from the following five questions. $\mathbf{8 \times 3 = 24}$

1. **a)** Prove that a graph is bipartite if and only if it does not contain any odd cycle.

b) Draw a graph whose incidence matrix is given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

c) Justify whether it is possible or not to draw a graph with 12 vertices having 13 edges

$\mathbf{5+2+1}$

2. **a)** Show that any simple graph with n vertices has at least two vertices of same degree**.**

b) Suppose the maximum degree of a tree T is 4. Let $n_i(T)$ denote the number of vertices having degree i. If $n_1(T) = 20$ and $n_2(T) = n_3(T) = n_4(T)$, find the number of vertices of T.

c) Define diameter and centre of a graph. $\mathbf{3+3+2}$

3. **a)** Prove thata graph G has a spanning tree if and only if G isconnected.

b) If G is a connected graph with each vertex of even degree then prove that $G$is Eulerian. $\mathbf{3+5}$

4. **a)** Give an example of a connected planar graph with v vertices, e edges and $k$components such that $e = 3v - 6k$.

b) Is the graph $K_{3,3}$ planar? Justify your answer.

c) Let G be a simple planar graph. Prove that there exists a vertex v in G such that $deg(v) \leq 5$.

d) Give an example of a Hamiltonian graph which is not Eulerian. $\mathbf{2+2+3+1}$

5. **a)** For any simple graph *G*, prove that $\chi(G) \leq \triangle(G) + 1$. Give an example of a simple graph where the equality holds.

b) Let C_n be a cycle of length $n, n \geq 2$. Prove that $\chi_{C_n}(k) = (k-1)^n + (-1)^n(k-1)$.

$\mathbf{(3+1)+4}$

<u>Group B – Field Theory</u>

Answer any two questions from the following three questions. $\mathbf{8 \times 2 = 16}$

6. a) Give an example of two non-isomorphic fields which are isomorphic as vector spaces.

b) Consider the field $K(x)$ and let $u = \frac{x^3}{x+1}$. Show that $K(x)$ is a simple extension of $K(u)$.

Also find $[K(x): K(u)]$.

c) Construct a field with 9 elements. $\mathbf{2+3+3}$

7. a) Give an example (with reason) of a field extension which is not a Galois extension.

b) If K is an infinite field, then prove that $K(x)$ is Galois over K.

c) For any finite group G, can you find a Galois extension with the Galois group isomorphic to G?

Justify your answer. $\mathbf{2+3+3}$

8. a) Find the splitting field of the polynomial $(x^3-5)(x^2-4)$ over $\mathbb{Q}$.

b) Give an example (with reason) of an algebraic field extension which is not separable.

c) For any finite field F, prove that the group $(F^*,\cdot)$ is cyclic. $\mathbf{2+3+3}$

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