M.SC. FOURTH SEMESTER EXAMINATIONS, 2021

Subject: Mathematics **Course ID: 42152**

Course Code: Math-402C **Course Title: Graph Theory & Field Theory**

Full Marks: 40 **Time: 2 Hours**

The figures in the margin indicate full marks

Notations and symbols have their usual meaning.

Group A- Graph Theory

Answer any two questions from the following three questions. $\mathbf{8 \times 2 = 16}$

1. **a)** Prove that a graph is bipartite if and only if it does not contain any odd cycle.

   b) Draw a graph whose incidence matrix is given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

   c) Justify whether it is possible or not to draw a graph with 12 vertices having 13 edges

$\mathbf{5 + 2 + 1}$

2. **a)** Prove that a graph G has a spanning tree if and only if G isconnected.

   b) If G is a connected graph with each vertex of even degree then prove that $G$is Eulerian. $\mathbf{3 + 5}$

3. **a)** Give an example of a connected planar graph with v vertices, e edges and $k$components such that $e = 3v - 6k$.

   b) Is the graph $K_{3,3}$ planar? Justify your answer.

   c) Let C_n be a cycle of length $n, n \geq 2$. Prove that$\chi_{C_n}(k) = (k - 1)^n + (-1)^n(k - 1)$.

$\mathbf{2 + 2 + 4}$

Group B – Field Theory

Answer any three questions from the following five questions. $\mathbf{8 \times 3 = 24}$

4. a) Prove that $\mathbb{Q}(\pi) \cong \mathbb{Q}(e)$ as fields.

b) Give an example of an algebraic extension which is not of finite dimensional.

c) Let Kbe a field and $f \in K[x]$ be an irreducible polynomial of degree n. Then show that there exists a unique (up to isomorphism which is the identity on K) extension $F = K(u)$ of K such that $[F{:}K] = n, u \in F$ is a root of F. $\mathbf{2 + 2 + 4}$

5. a) IfF is an algebraic extension field of E and E is an algebraic extension field of K, then prove that F is an algebraic extension of K.

b) Let F be a field extension over K and $u, v \in F$ such that v is algebraic over $K(u)$ and v is transcendental over K. Show that u is algebraic over $K(v)$.

c) Find the dimension of the field extension $\mathbb{Q}(\sqrt{11}, \sqrt{3}, \sqrt{9})$ over $\mathbb{Q}(\sqrt{12})$ by exhibiting its basis. **$3 + 3 + 2$**

6. a) Give an example of two non-isomorphic fields which are isomorphic as vector spaces.

b) Consider the field $K(x)$ and let $u = \frac{x^3}{x+1}$. Show that $K(x)$ is a simple extension of $K(u)$. Also find $[K(x): K(u)]$.

c) Construct a field with 9 elements. **$2 + 3 + 3$**

7. a) Give an example (with reason) of a field extension which is not a Galois extension.

b) If K is an infinite field, then prove that $K(x)$ is Galois over K.

c) For any finite group G, can you find a Galois extension with the Galois group isomorphic to G? Justify your answer. **$2 + 3 + 3$**

8. a) Find the splitting field of the polynomial $(x^3 - 5)(x^2 - 4)$ over $\mathbb{Q}$.

b) Give an example (with reason) of an algebraic field extension which is not separable.

c) For any finite field F, prove that the group $(F^*, \cdot)$ is cyclic. **$2 + 3 + 3$**

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