UNDERGRADUATE FOURTH SEMESTER (HONOURS) EXAMINATIONS, 2022

Subject: Mathematics **Course ID: 42111**

Course Code: SH/MTH/401/C-8

Course Title: Riemann Integration and Series of Functions

Full Marks: 40 **Time: 2 Hours**

The figures in the margin indicate full marks

The notations and symbols have their usual meaning

1. **Answer *any five* of the following questions:** $(\mathbf{2 \times 5 = 10})$

a) Find the set of all x for which the power series $\sum_{n=1}^{\infty} \frac{n}{(2n+1)^2}(x-3)^{3n}$ converges.

b) Let $f_n(x) = x^{\frac{1}{n}}$ for $x \in [0,1]$. Does $\lim_{n\to\infty} f_n(x)$ exist for all $x \in [0,1]$? Does $\{f_n(x)\}$ converge uniformly on [0,1]?

c) Find $\lim_{x\to 0} \frac{1}{x}\int_x^{2x} e^{-t^2}\, dt$.

d) Which of the following is not true? Justify your answer.

(i) Every Riemann integrable function is monotone increasing.

(ii) Every monotone function is Riemann integrable.

e) Which one of the following improper integrals is convergent? Justify.

(i) $\int_1^2 \frac{\sqrt{x}}{\log x} dx$ (ii) $\int_0^1 \frac{1}{\sqrt{x}(x+1)} dx$

f) Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function. If $\int_0^x f(2y)dy = \frac{x}{\pi}\sin(\pi x), \forall x \in \mathbb{R}$, find $f(2)$.

g) Which of the following is true? Justify your answer.

(i) $f(x)$ is not continuous on a closed interval but $f(x)$ may possess a primitive there.

(ii) $f(x)$ is continuous on a closed interval but $f(x)$ may not possess a primitive or anti-primitive there.

h) Expand f in a Fourier series in the interval $[-\pi, \pi]$, where

$$f(x) = \begin{cases} 1, & -\pi < x \le 0 \\ -2, & 0 < x \le \pi \end{cases}.$$

2. **Answer *any four* of the following questions:** $(\mathbf{5 \times 4 = 20})$

a) Define the Darboux integral. Let $f: [0,1] \to \mathbb{R}$ and $g: [0,1] \to \mathbb{R}$ be two functions defined by

$$f(x) = \begin{cases} \frac{1}{n}, & if x = \frac{1}{n}, n \in \mathbb{N} \\ 0, & otherwise \end{cases} \text{ and } g(x) = \begin{cases} n, & if x = \frac{1}{n}, n \in \mathbb{N} \\ 0, & otherwise. \end{cases}$$

Verify whether f and g are Darboux integrable or not on [0,1].

b) Let $p > 1$ and $q < 3p - 2$. Is the series $\sum \frac{1}{n^p + n^q x^2}$ uniformly convergent for all values of x? Justify. Is the series differentiable term by term? Explain.

c) If f is Darboux integrable in the interval $[a,b]$, then find

$$\lim_{n\to\infty} \int_a^b f(x) \cos nx dx.$$

d) Let f be continuous on [0,1] and $\int_0^1 x^n f(x) dx = 0$, for $n = 0, 1, 2, 3, \ldots$ Find $f(x)$ with proper justification.

e) Examine the convergence of the integral: $\int_0^1 x^{m-1} (1-x)^{n-1} \log x \, dx, m > 0, n > -1.$

f) Find the Fourier series for the function

$$f(x) = \begin{cases} \frac{\pi}{3}, & \text{for } 0 < x < \frac{\pi}{3} \\ 0, & \text{for } \frac{\pi}{3} < x < \frac{2\pi}{3} \\ -\frac{\pi}{3}, & \text{for } \frac{2\pi}{3} < x < \pi. \end{cases}$$

Also, find the sum of the series when $x = \pi, \frac{2\pi}{3}$.

3. Answer *any one* of the following questions: $(\mathbf{10 \times 1 = 10})$

a) **(i)** Let $f_n(x) = nx(1-x^2)^n$ and $f(x) = 0$, for $0 \le x \le 1$. Does $\{f_n(x)\}$ converge point wise to f? Justify. Does $\{f_n(x)\}$ converge uniformly to f? Justify. Is $\lim_{n\to\infty} \int_0^1 f_n \, dx \neq \int_0^1 f dx$? Verify.

(ii) Define the Riemann integral. Prove that if f is Riemann integrable on $[a, b]$ then f is bounded and Darboux integrable on $[a, b]$. 5+5

b) **(i)** Let $f: [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0, & \text{when, } x \text{ is irrational} \\ \frac{1}{q}, & \text{when } x = \frac{p}{q}, \text{ where, } p, q \text{ are positive integers prime to each other.} \end{cases}$$

Then verify whether f is Darboux integrable or not on [0,1].

(ii) Let $f: [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and P be a partition of $[a, b]$ with $\|P\| = \delta$. If P_K is a refinement of P with K additional points, then show that

$$0 \le U(P, f) - U(P_K, f) \le (M - m)K\delta.$$

(iii) Is the power series

$$x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \cdots$$

uniformly convergent on [-1,1]? Verify. 3+3+4

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