M.Sc.-III/Mathematics-304ME/18

M.Sc. 3rd Semester Examination, 2018

MATHEMATICS

(OTA-I)

Paper : 304ME

Course ID : 32154

Time: 2 Hours **Full Marks: 40**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and Symbols have their usual meanings.

Answer *any five* questions: 5×8=40

1. (a) Define adjoint operator of an operator $T: X \to Y$, where X, Y are normed linear spaces. Also find the adjoint operator of an operator $T: X \to Y$, where $T(x) = y$ and $g(y) = k$ (a constant), $\forall\, x \in X$ *and* $\forall\, g \in Y'$.

(b) If $T_1, T_2, T_3 \in B\,(X, Y)$ and α, β be any scalars, then show that

$(\alpha\, T_1 + \beta\, T_2\, T_3)^X = \alpha\, T_1^X + \beta\, T_3^X\, T_2^X$. 2+3+3=8

2. (a) Let X, Y be normed linear spaces. Then show that the operator $T: X \to Y$ is compact *iff* it maps every bounded sequence $\{x_n\}_n$ in X onto a sequence $\{Tx_n\}_n$ in Y which has a convergent subsequence.

(b) Show that the operator T: $l^2 \to l^2$, defined by $(Tx)(n) = \frac{x_n}{2^n}$, where $x = \{x_n\}_n$ is compact.

(c) Does there exist a compact linear operator $T: l^\infty \to l^\infty$ which is onto. 3+3+2=8

3. (a) Show that any totally bounded subset of a complete metric space is relatively compact.

(b) Let X, Y be two normed linear spaces. Show that the adjoint operator of a compact linear operator is compact. 3+5=8

4. (a) Let $T: X \to X$ be a compact linear operator on a normed linear space X and let $\lambda \neq 0$. Then show that $T^X f - \lambda f = g$ has a solution *iff* g is such that $g(x) = 0$. $\forall\, x \in X$ satisfying $Tx - \lambda x = \theta$.

(b) If T be a compact linear operator on a normed linear space X, then show that for every $\lambda \neq 0$, null space of T_λ is finite dimensional. 5+3=8

5. (a) Let X be a complex inner product space. If $T: X \to X$ is a bounded linear operator such that $\langle Tx, x \rangle = 0, \forall\, x \in X$, then show that $T = 0$.

(b) Give an example of an operator 'T' on a normed linear space X such that $\langle Tx, x \rangle = 0$, $\forall\, x \in X$ but $T \neq 0$.

(c) Show that a bounded linear operator T on a complex Hilbert space is unitary if T is isometric and onto. 3+2+3=8

6. (a) Define positive operator.

(b) Let $\{T_n\}_n$ be a sequence of bounded self adjoint linear operator on a complex Hilbert space H such that $T_1 \leq T_2 \leq \cdots \leq T_n \ldots \leq K$, where K is a bounded self adjoint operator on H. If any T_i commutes with every T_j and K, then show that $\{T_n\}_n$ is strongly operator convergent to a bounded self adjoint operator. 1+7=8

7. (a) Let P_1, P_2 be two projections on a Hilbert space H. Then show that

(i) P_1P_2 is a projection on H *iff* $P_1P_2 = P_2P_1$.

(ii) $P_2 - P_1$ projects H onto $[P_1(H)]^{\perp} \cap [P_2(H)]$, if $P_2 - P_1$ is a projection on H.

(b) Show that for any projection P on a Hilbert space H, $0 \leq \|P\| \leq 1$. 3+3+2=8

8. (a) Let H be a Hilbert space. If $T: H \to H$ is self adjoint then show that $\langle Tx, x\rangle$ is real $\forall x \in H$.

(b) Let S and T be two bounded linear operators on a Hilbert space H. If S is unitary equivalent to T and T is self adjoint. Then show that S is self adjoint.

(c) Let $T: C^2 \to C^2$ defined by $Tx = (\xi_1 + i\xi_2, \xi_1 - i\xi_2)$ where $x = (\xi_1, \xi_2)$. Find Hilbert adjoint T*.

(d) Let $P_i: H \to H$ be a projection on a Hilbert space $H, i = 1,2, \ldots, n$. If $P_1 + P_2 + \cdots + P_n$ be a Projection, then show that $\|P_1x\|^2 + \|P_2x\|^2 + \cdots + \|P_nx\|^2 \leq \|x\|^2$ for all $x \in H$.

1+2+3+2=8

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