POSTGRADUATE THIRD SEMESTER EXAMINATIONS, 2021

Subject: Mathematics **Course ID: 32151**

Course Code: Math-301C **Course Title: Functional Analysis**

Full Marks: 40 **Time: 2 Hours**

The figures in the margin indicate full marks

Notations and symbols have their usual meaning

Answer *any* *Five* of the following questions: **(8×5=40)**

1. (a) If *f* is a linear functional defined on *E* and *E* is of finite dimension, prove that *f* is bounded. Is the linear functional *f* continuous? Justify your answer. 4+1

   (b)Prove that in an inner product space *X*, $|(x,y)| \leq \|x\| \, \|y\|$. Find also the necessary and sufficient conditions for the equality. 3

2. (a) Give an examplewith justification of a Banach space which is not a Hilbert space. 3

   (b) Let E, Fbe normed linear spaces. Prove that for $A \in L(E,F)$,

   $\|A\| = \sup\{\|Ax\| : \|x\| \leq 1\}$. 3

   (c) What is meant by a function space? Give an example of such a space. 1+1

3. (a) Prove that in a normed linear space*E*, if $\{x_n\}$ is a Cauchy sequence of elements and $\{\lambda_n\}$ is a Cauchy sequence of scalers, then $\{\lambda_n x_n\}$is also a Cauchy sequence in *E*. 3

   (b) If *M* and *N* are closed subspaces of a Hilbert space *H* such that $M \perp N$, prove that the subspace *M*+*N* is closed. 4

   (c) Find the norm of the identity linear transformation. 1

4. (a) Let $T: \mathbb{R}^4 \to \mathbb{R}^2$ be a linear transformation defined by

$$T\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a+d \\ b+c \end{bmatrix}$$

for all $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4$.

Show that *T* is onto but not one-one. 2+2

(b) Prove that the closure of a convex subset of a normed linear space is also convex. 3

(c) Show that the norm function is continuous. 1

5. (a) If the dual space E^* of a normed linear space *E* is separable, prove that *E* is also separable. 5

(b) If *M* is a closed subspace of a Hilbert space *H*, then show that $M^{\perp\perp} = M$. 3

6. (a) Let *C* be a closed convex subset of a Hilbert space *H*. Prove that *C* contains a unique element of the smallest norm. 5

(b) Prove that on a finite dimensional linear space *X*, any norm $\|\,.\,\|_1$ is equivalent to any other norm $\|\,.\,\|_2$. 3

7. (a) Give an example with justification of a linear transformation which is not bounded. 3

(b)Prove that the series

$$\sum_{i=1}^{\infty} \alpha_i e_i$$

where $\alpha_1, \alpha_2, \ldots$are scalars is convergent if and only if the series $\sum_{i=1}^{\infty} |\alpha_i|^2$

converges. 3

(c) State Riesz representation theorem on a Hilbert space. 2

8. (a) State the Uniform Boundedness Principle on Banach spaces. 2

(b) Prove that the normed space *X* of all polynomials with norm defined by

$$\|x\| = \max_{j} |\alpha_j| \quad (\alpha_0, \alpha_1, \ldots \text{ are the coefficients of } x)$$

is not complete. 5

(c) Let *M* and *N* be two subspaces of a linear space *L* such that *L*=*M*+*N*. Write down a necessary and sufficient condition for $L = M \oplus N$. 1

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