M. SC. SECOND SEMESTER EXAMINATIONS, 2021

Subject: Mathematics **Course ID: 22153**

Course Code: Math-203C

Course Title: Calculus of Several Variables & Differential Geometry of Curves and Surfaces

Full Marks: 40 **Time: 2 Hours**

The figures in the margin indicate the full marks

Notations and symbols have their usual meaning

Group-A (Calculus of Several Variables)

Answer any three of the following questions. $\mathbf{8 \times 3 = 24}$

1. (i) Suppose $f: \mathbb{R}^n \to \mathbb{R}$ is a C^1-function on an open ball containing the point c. Then show that for any unit vector u, $D_u f(c)$ exists and $D_u f(c) = \nabla f(c) \cdot u$

   (ii) Let $V \subset \mathbb{R}^n$ be an open set and let $c \in V$. When a mapping $f: V \to \mathbb{R}^m$ is differentiable at c? 6+2

2. (i) If a function $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $c \in \mathbb{R}^n$, then show that f is continuous at c.

   (ii) Consider two functions $f: V \to \mathbb{R}^m$ and $g: V \to \mathbb{R}^m$, where $V \subset \mathbb{R}^n$ is an open set. Let f and g be both differentiable at $a \in V$. Show that the function $(f \cdot g): V \to \mathbb{R}$, given by $(f \cdot g)(c) = f(c) \cdot g(c),\ \forall\, c \in V$, is also differentiable at a, and

   $$\big(D(f \cdot g)\big)(a) = g(a).(Df)(a) + f(a).(Dg)(a)$$

   where "$\cdot$" denotes usual dot product of vectors and "." denotes matrix multiplication. 3+5

3. (i) Define Hessian of a function of $f: \mathbb{R}^n \to \mathbb{R}$..

   (ii) Suppose $f: \mathbb{R}^2 \to \mathbb{R}$ is defined on an open set $V \subset \mathbb{R}^2$, such that f_x, f_y and f_{xy} exist at every point of V, and f_{xy} is continuous at some point $(a, b) \in V$. Then show that f_{yx} exist at (a, b) and

   $$f_{xy}(a, b) = f_{yx}(a, b).$$

   (iii) Give an example of a function $f: \mathbb{R}^2 \to \mathbb{R}$ for which $f_{xy}(a, b) \neq f_{yx}(a, b)$ at some point $(a, b) \in \mathbb{R}^2$. 2+4+2

4. (a) Show that a linear transformation $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at each point of $\mathbb{R}^n$.

   (b) Let $A \subset \mathbb{R}^n$ and p be a limit point of A. Let $f : A \to \mathbb{R}^n$ be a function such that $\lim_{x \to p} f(x) = a$. Prove that $\lim_{x \to p} \|f(x)\| = \|a\|$.

   (c) State Inverse Function theorem for a function f on $\mathbb{R}^n$. 3+3+2

5. (a) Let $g: A \to \mathbb{R}^n$ be a continuously differentiable function, where $A \subset \mathbb{R}^n$ is open, and let $B = \{x \in A : \det g'(x) = 0\}$. Show that $g(B)$ has measure zero.

(b) What do you mean by partition of unity? 6+2

Group-B (Differential Geometry of Curves and Surfaces)

Answer any two of the following questions. $\mathbf{8 \times 2 = 16}$

6. (a) Show that any covariant tensor of second order can be expressed uniquely as the sum of a symmetric and a skew-symmetric tensor of the second order. 5+3

(b) Show that $R^l{}_{ijk} + R^l{}_{jki} + R^l{}_{kij} = 0$, where $R^l{}_{ijk}$ is the Riemann curvature tensor of type $(1,3)$.

7. (i) Let $\gamma(t)$ be a regular curve in $\mathbb{R}^3$. Then prove that its curvature is given by

$$\kappa = \frac{\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^3},$$

where dot denotes derivative with respect to t. 3+2+(1+2)

(ii) Find the curvature of the space curve $\gamma(\theta) = (2\cos\theta, 2\sin\theta, 3\theta)$.

(iii) When a space curve is said to be of unit speed. Check whether the following curve is of unit speed or not

$$\gamma(t) = \left(\frac{4}{5}\cos t, 1 - \sin t, -\frac{3}{5}\cos t\right)$$ 3+2+(1+2)

8. (i) Let $\sigma: U \to \mathbb{R}^3$ be a regular surface patch of a surface S and let $P \in S$. Let (u, v) be coordinates of P in U. Show that the tangent space $T_P S$ is a 2-dimensional vector subspace of $\mathbb{R}^3$ spanned by the vectors σ_u and σ_v.

(ii) Find the first fundamental form of the plane

$$\sigma(u, v) = \vec{a} + u\,\hat{p} + v\,\hat{q},$$

where $\vec{a}$ is a fixed vector and $\hat{p}$, $\hat{q}$ are two unit vectors perpendicular to each other.

(iii) State Frenet-Serret formulae for a space curve. 4+2+2

***