M.SC. SECOND SEMESTER EXAMINATIONS, 2021

Subject: Mathematics **Course ID: 22151**

Course Code: Math-201C **Course Title: Complex Analysis**

Full Marks: 40 **Time: 2 Hours**

<u>The figures in the margin indicate full marks</u>

Notations and symbols have their usual meaning.

Answer any five questions from the following eight questions. $\mathbf{8 \times 5 = 40}$

1. a) Find all the Mobius transformations which transform the circle $|z| \leq 1$ into the circle $|w| \leq 1$.

b) Find the maximum modulus of the function $f: D \rightarrow \mathbb{C}$ defined by $f(z) = z - 4i$ on D where $D: |z| \leq 1$.

c) Find the cross-ratio of $1, -1, i, -i$. $\mathbf{4+3+1}$

2. a) If $f: D \rightarrow \mathbb{C}$ where $D: |z| < 1$ is analytic on D and if f satisfies the condition $|f(z)| < 1, f(0) = 0$, then show that $|f(z)| < |z|$ and $|f'(0)| < 1$.

b) Suppose that a complex valued function f is analytic in some domain D of the complex plane which contains a segment of the real axis and whose lower half is the reflection of the upper half with respect to the axis. Prove that $f(x)$ is real at each point on the segment if $\overline{f(z)} = f(\bar{z})$.

c) Show that $\lim_{z\rightarrow\infty} f(z) = \lim_{z\rightarrow 0} \frac{1}{f\left(\frac{1}{z}\right)}$. $\mathbf{4+3+1}$

3. a) Use the definition of contour integration to evaluate $\int \pi e^{\pi \bar{z}} dz$ over the positively oriented contour which is a square with vertices at $0, 1, 1+i, i$.

b) Suppose f is a continuous function on a domain D such that f has an antiderivative there. Show that the integrals of f around the closed contours lying entirely in D all have value zero.

c) Let $C: |z| = 1$ (positively oriented). Evaluate $\int_C \frac{dz}{z^2+2z+2}$. $\mathbf{3+3+2}$

4. a) Use Cauchy integral formula to evaluate $\int_C \frac{dz}{(z^2+9)^2}$ where C is the negatively oriented circle $|z - 2i| = 2$.

b) State and prove Fundamental theorem of algebra. $\mathbf{3+1+4}$

5. a) Derive the Taylor series representation of $\frac{1}{z-i}$ with respect to $z_0 = 1$.

b) Find the Maclaurin series expansion of $\frac{z^3}{z^4+16}$.

c) Deduce any two possible Laurent series expansions of $\frac{1}{z^3(1-z)}$. **$3+2+3$**

6. **a)** State Cauchy residue theorem. Using Laurent's theorem and Cauchy residue theorem evaluate $\int \frac{dz}{z^2 sinhz}$ over the positively oriented contour $|z| = \pi/2$.

b) Using residues evaluate $\int \frac{(3z^3+2)dz}{(z-1)(z^2+9)}$ over the positively oriented contour $|z| = 4$. **$1+4+3$**

7. **a)** State and prove Casorati-Weierstrass theorem.

b) Find the singularities with their nature of the function $\frac{1}{\sin\left(\frac{\pi}{z}\right)}$. **$1+4+3$**

8. a) State the Argument principle. Find the winding number of the image of the positively oriented contour $|z| = 1$ under the transformation $w = \frac{z^3+2}{z}$ with respect to the origin.

b) Use residues to evaluate the improper integral $\int_0^\infty \frac{x^2 dx}{(x^2+1)(x^2+4)}$. **$1+2+5$**

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