B.Sc. 2nd Semester (Honours) Examination, 2019

MATHEMATICS

(Real Analysis)

Paper : 201/C-3

Course ID : 22111

Time: 2 Hours **Full Marks: 40**

The figures in the right hand side margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

1. Answer *any five* questions: 2×5=10

(a) Is the least upper bound axiom true for the set of rational numbers? Justify it.

(b) Give example of two uncountable sets A and B in $\mathbb{R}$ (the set of real numbers) such that $A \cap B$ is countable.

(c) Find the derived set of the set $S = \left\{\frac{1}{3^n} + \frac{1}{4^m} : m, n \in \mathbb{N}\right\}$.

(d) Find the upper and lower limit of the sequence $\left\{(-1)^n + \sin\frac{n\pi}{4}\right\}$.

(e) Prove that a convergent sequence is bounded, but the convers may not be true. 1+1=2

(f) Examine if the set $S = \{x \in \mathbb{R} : \sin x = 0\}$ is closed in $\mathbb{R}$.

(g) Give an example to show that arbitrary union of compact sets may not be compact.

(h) Examine the convergence of the series $\frac{1}{1+a^2} - \frac{1}{2+a^2} + \frac{1}{3+a^2} - \cdots\cdots$, a is real number.

2. Answer *any four* questions: 5×4=20

(a) (i) Show that the derived set of a bounded set is bounded.

(ii) If $y > 0$ show that there exists $n \in \mathbb{N}$ (the set of natural numbers) such that $\frac{1}{2^n} < y$. 3+2=5

(b) (i) Let E be a bounded set of real numbers and M be the supremum of E. If $M \notin E$, show that M is a limit point of E.

(ii) Let $\{a_n\}$ be a sequence of positive real numbers with $\overline{\lim}\ a_n^{1/n} = r$. Prove that $\sum_{n=1}^{\infty} a_n$ is convergent if $r < 1$ and divergent if $r > 1$. 2+3=5

(c) Show that a bounded sequence $\{u_n\}$ is convergent if and only if $\underline{\lim}\, u_n = \overline{\lim}\, u_n$.

(d) (i) Prove that an absolutely convergent series can be expressed as difference of two convergent series of positive real numbers.

(ii) Applying Cauchy's general principle of convergence show that the series
$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} + \cdots$ is convergent. 3+2=5

(e) Show that every infinite bounded subset of $\mathbb{R}$ has at least one limit point in $\mathbb{R}$.

(f) (i) Prove that $\lim_{n\to\infty} \frac{(\lfloor n)^{1/n}}{n} = 1/e$.

(ii) Prove that the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is a convergent. 2+3=5

3. Answer *any one* question: 10×1=10

(a) (i) Prove that every bounded sequence of real numbers has a convergent subsequence.

(ii) Examine the convergence of the following series:

$1+\frac{(\alpha+1)}{(\beta+1)}+\frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)}+\frac{(\alpha+1)(2\alpha+1)(3\alpha+1)}{(\beta+1)(2\beta+1)(3\beta+1)}+\cdots\cdots$, whre α, β are $+v_e$ real numbers.

(iii) If $\sum a_n$ be a convergent series of positive and non-increasing terms show that $\lim_{n\to\infty} na_n = 0$. 4+3+3=10

(b) (i) Prove that the derived set of a set is always closed set.

(ii) Show that $\lim_{n\to\infty} \frac{1}{n}\{(n+1)(n+2)\cdots(n+n)\}^{1/n} = \frac{4}{e}$.

(iii) Prove that $\mathbb{R}$ is not compact.

(iv) Give an example of an open cover of the set (0, 5] which does not have a finite sub-cover. 3+3+2+2=10

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