B.Sc. Semester-II (Honours) Examination, 2018

MATHEMATICS

Subject Code : 22101 **Course Code : SHMATH/201/C-3**

Course Title : Real Analysis

Time: 2 Hours **Full Marks: 40**

The figures in the right hand side margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notation and Symbols have their usual meanings.

1. Answer *any five* questions: 2×5=10

(a) Show that $\lim_{n\to\infty} \frac{1}{n}\left\{1 + 2^{\frac{1}{2}} + 3^{\frac{1}{3}} + \cdots + n^{\frac{1}{n}}\right\} = 1$. 2

(b) Let S be a non-empty bounded subset of $\mathbb{R}$. Let $a > 0$ and let $aS = \{as | s \in S\}$. Then prove that $\inf.(aS) = a(\inf. S)$. 2

(c) If $\{x_n\}_n$ converges to l, then show that $\{|x_n|\}_n$ converges to $|l|$. 2

(d) Is $\{1 + (-1)^n\}_n$ a Cauchy sequence? Give reason. 2

(e) If the series $\sum_{n=1}^{\infty} u_n$ converges then show that $\lim_{n\to\infty} u_n = 0$. 2

(f) Show that the set $\mathbb{N}$ (the set of all natural numbers) has no limit point. 2

(g) Show that the infinite series $\frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots$ converges for $p > 0$. 2

(h) Define conditionally convergent series with an example. 2

2. Answer *any four* questions: 5×4=20

(a) Show that the sequence $\left\{\left(1 + \frac{1}{n}\right)^n\right\}$ is convergent. 5

(b) If a sequence $\{x_n\}_n$ converges to l, then every subsequence of $\{x_n\}_n$ also converges to l. Also show that the converse of the above is not true in general. 3+2=5

(c) Define enumerable set. Show that union of two enumerable sets is enumerable again. 1+4=5

(d) (i) Show that the series $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$ is not convergent.

(ii) Prove that every absolutely convergent series is convergent.

(iii) Does the divergence of $\sum_{n=1}^{\infty} |u_n|$ imply divergence of $\sum_{n=1}^{\infty} u_n$? Justify your answer. 1+2+2=5

(e) Show that the series $\sum \frac{1}{n^p}$ is convergent for $p > 1$. 5

(f) Prove that the set (0, 1) is uncountable. 5

3. Answer *any one* question: 10×1=10

(a) (i) Define closure of a set. Let A and B be two non-empty subsets of $\mathbb{R}$. Does $\overline{A} \cap \overline{B} \subset \overline{A \cap B}$ hold always? Justify your answer.

(ii) Let $\{a_n\}$ and $\{b_n\}$ be two bounded sequences of non-negative real numbers. Then prove that $(\underline{\lim}\, a_n).(\underline{\lim}\, b_n) \le \underline{\lim}(a_n\, b_n)$. Give an example to show that strict inequality may happen in this case.

(iii) Show that the infinite series $\sum \frac{1}{n^2} \cos nx$ is absolute convergent for all x. (1+2)+(3+2)+2=10

(b) (i) Prove that a real number c is a limit point of a set A iff every neighbourhood of c contains infinitely many points of A.

(ii) If $a_1 = \sqrt{3}$ and $a_{n+1} = \sqrt{3a_n}$, then show that $\lim_{n \to \infty} a_n = 3$.

(iii) Test the convergence of the series $1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \cdots$ for $0 < x \le 1$. 4+3+3=10

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B.Sc. Semester-II (Honours) Examination, 2018

MATHEMATICS

Subject Code : 22102 **Course Code : SHMATH/202/C-4**

Course Title : Differential Equations and Vector Calculus

Time : 2 Hours **Full Marks : 40**

The figures in the right hand side margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and Symbols have their usual meanings.

1. Answer *any five* questions: 2×5=10

(a) State Picard's theorem on existence and uniqueness of solution of an initial value problem (IVP) $\frac{dy}{dx} = f(x,y),\ y(x_0) = y_0$. 2

(b) If $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \vec{0}$, show that the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar. 2

(c) Show that $f(x,y) = x \sin y + y \cos x$ satisfies the Lipschitz condition over the rectangular region $R = \{(x,y): |x| \le a, |y| \le b\}$. Also find the Lipschitz constant. 2

(d) If y_c is the general solution of the differential equation (DE) $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ and y_p is any particular solution of the DE $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$, then show that $(y_c + y_p)$ is the general solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$. 2

(e) Show that the function e^{ax}, e^{bx} and e^{cx} are linearly independent, where $a \ne b \ne c$. 2

(f) Evaluate $\int_1^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt$, where $\vec{r} = 2t^2\hat{\imath} + t\hat{\jmath} - 3t^2\hat{k}$. 2

(g) Determine whether $x = 0$ is a singular point or an ordinary point of the DE:

$x^2\frac{d^2y}{dx^2} + 3(x^2 + 2x)\frac{dy}{dx} - 2y = 0$ 2

(h) What do you mean by Phase Plane of a system of equations? 2

2. Answer *any four* questions: 5×4=20

(a) Define Wronskian of two differentiable functions $f(x)$ and $g(x)$.

Show that the value of the Wronskian $W(f_1, f_2)$ of two solutions f_1 and f_2 of the differential equation $a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0, a_1(x) \ne 0$ and a_0, a_1, a_2 are continuous functions of x, on some interval $c \le x \le d$ is either identically equal to zero on $c \le x \le d$ or never zero on $c \le x \le d$. 1+4=5

(b) (i) Define equilibrium point of a plane autonomous system.

(ii) Determine the nature of the equilibrium point of the linear system: $\left.\begin{array}{l}\frac{dx}{dt}=x\\ \frac{dy}{dt}=y\end{array}\right\}$. Also sketch the corresponding phase portrait in the phase plane. 1+4=5

(c) Applying the method of variation of parameter, solve the differential equation

$$\frac{d^2y}{dx^2}+4y=\sec^2 2x.$$ 5

(d) Show that the equation $x^3\frac{d^3y}{dx^3}-6x\frac{dy}{dx}+12y=0$ has three linearly independent solutions of the form $y=x^m$. Hence solve it. 5

(e) Show that the necessary and sufficient condition for a vector $\vec{r}=\vec{f}(t)$ to have a constant direction is $\vec{f}\times\frac{d\vec{f}}{dt}=\vec{0}$. 5

(f) Solve $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+4y=\cos 4x$ by the method of undetermined coefficients. 5

3. Answer *any one* question: 10×1=10

(a) (i) Given that $y=x$ is a solution of the differential equation (DE):

$$x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+4y=0\ (x\neq 0),$$

find a solution linearly independent to $y=x$ by reducing the order of the given differential equation (DE).

Hence obtain the general solution of given DE.

(ii) By reducing normal form solve $\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+(4x^2-3)y=e^{x^2}$. 6+4=10

(b) (i) Obtain the power series solution of

$$2x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-(x+1)y=0$$

about the regular singular point $x=0$.

(ii) If $\vec{r}=a\cos t\,\hat{\imath}+a\sin t\,\hat{\jmath}+at\tan\alpha\,\hat{k}$, then find the values of $\left|\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right|$ and $\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right]$.

6+4=10

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B.Sc. Semester-II (Honours) Examination, 2018

MATHEMATICS

Subject Code : 22103 **Course Code : SHMATH/203/GE-2**

Course Title : Real Analysis

Time: 2 Hours **Full Marks: 40**

The figures in the right hand side margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and Symbols have their usual meanings.

1. Answer *any five* questions: 2×5=10

(a) Prove that a finite set has no limit point. 2

(b) Find $\lim_{n\to\infty} \frac{\sin n}{n}$. 2

(c) Define limit points of a set of real numbers and find limit points of $\left\{-1+\frac{1}{n} : n \in N\right\}$. 2

(d) When a set of real numbers is said to be compact? State Heine-Borel theorem. 2

(e) Show that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$. 2

(f) Give examples of divergent sequences $\{u_n\}$ and $\{v_n\}$ such that the sequence $\{u_n v_n\}$ is convergent. 2

(g) Test the convergence of the series whose nth term is $\frac{1}{\sqrt{n}}$. 2

(h) Test the infinite series $\frac{1}{2\cdot 3\cdot 4} + \frac{3}{3\cdot 4\cdot 5} + \cdots$, for convergence. 2

2. Answer *any four* questions: 5×4=20

(a) Show that the union of an arbitrary collection of open sets in $\mathbb{R}$ is an open set. What about the case for arbitrary intersection? Justify your answer. 3+2=5

(b) (i) Define Archimedean property of $\mathbb{R}$.

(ii) If $x \in \mathbb{R}$ and $x > 0$, then prove that there exists $k \in \mathbb{N}$ such that $0 < \frac{1}{k} < x$.

(iii) If $x \in \mathbb{R}$ and $x > 0$, then prove that there exists $k \in \mathbb{N}$ such that $-1 \leq x < k$. 1+1+3=5

(c) (i) Show that the union of two bounded sets is bounded.

(ii) State Bolzano-Weirstrass theorem for set and verify it for the set $\left\{1+\frac{1}{n} : n \epsilon N\right\}$. 2+(1+2)=5

(d) When a sequence is said to be convergent? Show that ''every convergent sequence is bounded'' but converse is not true. 1+3+1=5

(e) Define limit superior and limit inferior of a sequence of real numbers. Find the limit superior and limit inferior of the sequence whose n-th term S_n is given by $S_n = (-1)^n\left(1+\frac{1}{2n}\right)$. Also find a subsequence of this sequence that converges to the limit superior. 1+1+1+1+1=5

(f) (i) If $\sum_{n=1}^{\infty} u_n$ is a convergent series of positive reals, then prove that the series $\sum_{n=1}^{\infty}\frac{u_n}{n}$ is convergent.

(ii) Show that the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ is convergent. 2+3=5

3. Answer *any one* question: 10×1=10

(a) (i) Show that the open interval (0,1) is uncountable.

(ii) Show that every compact subset of $\mathbb{R}$ is bounded.

(iii) If $\lim_{n\to\infty} x_n = l$ and $\lim_{n\to\infty} y_n = m$, then prove that $\lim_{n\to\infty}\frac{x_n}{y_n}=\frac{l}{m}$, provided $m \neq 0$. 3+3+4=10

(b) (i) State Cauchy's Criteria of convergence of a sequence of real numbers.

A sequence $\{x_n\}$ is defined by

$$x_n = \sqrt{2x_{n-1}},\ (n>1),\ x_1=\sqrt{2}\ ;$$

show that the sequence is monotonic increasing and bounded above. Find the limit of the sequence.

(ii) State D'Alembert's Ratio test.

Prove that the series

$\frac{3}{1\cdot 2}-\frac{5}{2\cdot 3}+\frac{7}{3\cdot 4}-\frac{9}{4\cdot 5}+\cdots$ conditionally converges. (2+1+1+1)+(2+3)=10

B.Sc. Semester-II (Programme) Examination, 2018

MATHEMATICS

Subject Code : 22104 **Course Code : SPMATH/201/C-1B**

Course Title : Real Analysis

Time: 2 Hours **Full Marks: 40**

The figures in the right hand side margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notation and Symbols have their usual meanings.

1. Answer *any five* questions: 2×5=10

(a) Define neighbourhood of a point in ℝ. Give an example. 2

(b) Examine whether the set $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ is bounded. 2

(c) Give an example of a sequence which is bounded but not convergent. 2

(d) Show that the sequence $\left\{\frac{n+1}{n}\right\}_n$ is decreasing sequence. 2

(e) If A be an open set and B is closed such that $B \subset A$, then prove that $A - B$ is open. 2

(f) Define derived set of a set of reals. Find the derived set of the set $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$. 2

(g) State the Bolzano-Weirstrass theorem for a sequence in Real Number. 2

(h) Use Comparison test to prove that the series $\frac{1}{2^2} + \frac{\sqrt{2}}{3^2} + \frac{\sqrt{3}}{4^2} + \ldots$ is convergent. 2

2. Answer *any four* questions: 5×4=20

(a) (i) Define an enumerable set.

(ii) Show that the set $\mathbb{Z}$ is enumerable. 2+3=5

(b) (i) Let A and B be two bounded sets and let $S = \{a + b : a \in A, b \in B\}$. Prove that S is also bounded.

(ii) Hence show that $\sup A + \sup B = \sup S$. 4+1=5

(c) State D' Alembert's Ratio test for a series of (+)ve terms and also test the convergence of the series

$$\frac{1}{\lfloor 1} + \frac{2^2}{\lfloor 2} + \frac{3^2}{\lfloor 3} + \ldots + \frac{n^2}{\lfloor n} + \ldots$$

2+3=5

(d) Show that $\left\{\left(1 + \frac{1}{n}\right)^n\right\}_n$ converges. 5

(e) Show that the alternating series $\sum_{1}^{\infty}(-1)^{n+1}\frac{1}{n}$ is convergent by Leibnitz's test. 5

(f) Show that $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent. 5

3. Answer *any one* question: 10×1=10

(a) (i) Explain that a finite set has no limit point.

(ii) State the Archimedean Property of $\mathbb{R}$.

(iii) If $x \in \mathbb{R}$ and $x > 0$, then show that there exists a natural no. k such that $k-1 \leq x < k$.

3+2+5=10

(b) (i) Prove that the sequence $\{x_n\}$ where $x_n = \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}$ is a convergent sequence. What is your estimation of the value of the limit of this sequence?

(ii) Test the convergence of the series: $\frac{x}{1^2}+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\cdots$ 5+5=10

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