M.Sc.-I/Math-102C/18

M.Sc. 1st Semester Examination, 2018

MATHEMATICS

Paper : 102C

Course ID : 12152

Time: 2 Hours **Full Marks: 40**

The questions are of equal value.

The figures in the right hand side margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and unexplained symbols have their usual meanings.

Group–A

(Linear Algebra)

Answer *any three* questions. 8×3=24

1. (a) Let V be a finite dimensional vector space and let $\mathbb{B} = \{v_1, v_2, \ldots, v_n\}$ be a basis for V. Define the dual basis $\{f_1, f_2, \ldots, f_n\}$ of $\mathbb{B}$.

Deduce that for each linear functional f on V

$$f = \sum_{i=1}^{n} f(v_i) f_i$$

and also for each vector $v \in V$

$$v = \sum_{i=1}^{n} f_i(v) v_i$$

(b) Let $\mathbb{B} = \{v_1, v_2, v_3\}$ be the basis for $\mathbb{C}^3$ defined by $v_1 = (1, 0, -1), v_2 = (1, 1, 1)$, $v_3 = (2, 2, 0)$. Find the dual basis of $\mathbb{B}$. (1+2+2)+3=8

2. (a) Let V be the vector space of all polynomial functions over the field of real numbers. Let a and b be fixed real numbers ($a < b$) and let f be the linear functional on V defined by

$$f(p) = \int_a^b p(x)dx$$

If D is the differentiation operator on V, what is $D^t f$? ($D^t :=$ Transpose of D)

(b) Prove that the characteristic polynomial of any diagonalisable linear operator splits.

(c) Find a projection P which projects $\mathbb{R}^2$ onto the subspace spanned by (1, 2) along the subspace spanned by (1, –1). 2+3+3=8

3. (a) State and prove Caley Hamilton Theorem for linear operators.

(b) Find all possible Jordan forms for a 6×6 complex matrix with $x^2(1-x)^2$ as the minimum polynomial. (2+3)+3=8

4. (a) Let $V = C[0,1]$ and define $\langle f, g\rangle = \int_0^{\frac{1}{2}} f(t)g(t)dt$. Is this an inner product on V?

(b) Prove that an orthogonal set of non-zero vectors on an inner product space is linearly independent.

(c) Let $V = \mathbb{R}_2[t]$ be the space of all polynomials of degree at most 2 over $\mathbb{R}$. Let an inner product on V be defined by $\langle x, y\rangle = \int_0^1 x(t)\, y(t)dt$. Obtain an orthonormal basis from the basis $\{1, t, t^2\}$ for V over $\mathbb{R}$ using Gram-Schmidt orthonormalization process. 2+2+4=8

5. (a) Find the matrix representation of the Bilinear form $H: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$H\big((a_1, a_2), (b_1, b_2)\big) = 2a_1b_1 + 3a_1b_2 + 4a_2b_1 - a_2b_2$

w.r.t. the ordered basis $\mathbb{B} = \{(1,1), (1,-1)\}$.

(b) Define a real quadratic form on $\mathbb{R}^3$.

(c) Find the rank and signature of the real quadratic form $Q(x_1, x_2) = x_1^2 - 4x_1x_2 + x_2^2$.

(d) Investigate whether the following real quadratic form
$Q(x_1, x_2, x_3) = {x_1}^2 - 2x_1x_2 + x_2^2 + 4x_2x_3 + 5{x_3}^2$
is positive definite or not. 2+2+2+2=8

Group–B

(Calculus of Several Variables)

Answer *any two* questions. 8×2=16

6. (a) Let $f(x,y) = \frac{x^3-y^3}{x^2+y^2}$; $(x,y) \neq (0,0)$

$= 0$; $(x,y) = (0,0)$

Prove that the function is continuous at (0, 0).

(b) Show that the function

$f(x,y) = \frac{x|y|}{\sqrt{x^2+y^2}}$; $(x,y) \neq (0,0)$

$= 0$; $(x,y) = (0,0)$

is not differentiable at (0, 0).

(c) Find the gradient vector of $f(x,y,z) = x^2 - y^2 + 2z^2$ at (1, 1, 1) where $(x,y,z) \in \mathbb{R}^3$.
3+4+1=8

7. (a) If $f: \mathbb{R}^2 \to \mathbb{R}$ and $D_2 f = 0$, show that f is independent of the second variable. If $D_1 f = D_2 f = 0$, show that f is constant.

(b) Find f' for $f(x,y) = \int_a^{x+y} g$, where $g: \mathbb{R} \to \mathbb{R}$ is continuous.

(c) Evaluate the directional derivative of the scalar field $f(x,y,z) = x^2 + 2y^2 + 3z^2$ at (1, 1, 0) in the direction of $(\hat{\imath} - \hat{\jmath} + 2\hat{k})$. (1+2)+3+2=8

8. (a) Suppose $f: \mathbb{R}^n \to \mathbb{R}^n$ is differentiable and has a differentiable inverse $f^{-1}: \mathbb{R}^n \to \mathbb{R}^n$. Show that $(f^{-1})'(a) = \left[f'\left(f^{-1}(a)\right)\right]^{-1}$.

(b) Let the function $f: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (e^x \cos y, e^x \sin y)$. Show that $Det\, f'(x, y) \neq 0$ for all $(x, y) \in \mathbb{R}^2$ but f is not one-one.

(c) Define a real analytic function. 3+(2+1)+2=8