B.SC. FIRST SEMESTER (HONS.) EXAMINATION 2021

Subject: Mathematics **Course ID: 12112**

Course Title: Algebra **Course code: SH/MTH/102/C-2**

Full Marks: 40 **Time: 2 Hours**

<u>The figures in the margin indicate full marks</u>

Notations and symbols have their usual meaning.

1. Answer any five questions: $2 \times 5 = 10$

(a) Find the product of all the values of $\left(1+\sqrt{3}i\right)^{\frac{3}{4}}$.

(b) Prove that $3^{2n-1} + 2^{n+1}$ is divisible by 7, using principle of mathematical induction.

(c) Show that the set $S = \{(x, y, z) : x, y, z \in \mathbb{R}, 2x + y - z = 0\}$ is a subspace of $\mathbb{R}^3$.

(d) Use division algorithm to find integer u and v satisfying $30u + 72v = 12$.

(e) Let $f: \mathbb{Z} \to \mathbb{Z}$ be a mapping defined by $f(x) = x^2, x \in \mathbb{Z}$ and $g: \mathbb{Z} \to \mathbb{Z}$ be a mapping defined by $g(x) = 2x, x \in \mathbb{Z}$. Find fog and gof.

(f) If A be a set of nonzero integers and R be a relation defined on $A \times A$ such that $(a, b)R(c, d)$ iff $ad = bc$, then show that R is an equivalence relation.

(g) If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, then find the equation whose roots are $(\alpha + \beta)/\gamma$, $(\beta + \gamma)/\alpha, (\gamma + \alpha)/\beta$.

(h) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x_1, x_2, x_3) = (x_1 + 1, x_2 + 1, x_3 + 1)$. Examine whether T is a linear mapping or not.

2. Answer any four questions: $5 \times 4 = 20$

(a) (i) Show that if $n \geq 4$, then $n, n + 2$ and $n + 4$ cannot be all prime.

(ii) Use theory of congruence to find the remainder when $1! + 2! + 3! + + \cdots + 100!$ is divided by 15. 3+2=5

(b) (i) Use Cauchy- Schwarz inequality to prove that $(a + b + c + d)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}) > 16$, where a, b, c, d are positive real numbers, not all equal.

(ii) If x, y, z are positive real numbers and $x + y + z = 1$, prove that

$8xyz \leq (1 - x)(1 - y)(1 - z) \leq \frac{8}{27}$. 2+3=5

(c) Using De Moivre's theorem, prove that $\tan 5\theta = \frac{5 \tan\theta - 10\, tan^3\theta + tan^5\theta}{1 - 10\, tan^2\theta + 5\, tan^4\theta}$

(d) Solve the equation $x^4 - 6x^2 + 16x - 15 = 0$ by Ferrari's method.

(e) Determine the conditions for which the following system of equations has

(I) only one solution (II) no solution (III) many solutions

$$x + y + z = 1$$
$$x + 2y - z = b$$
$$5x + 7y + az = b^2. \qquad 5$$

(f) Verify Cayley-Hamilton theorem for the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix}$. Hence compute A^{-1}. 5

3. Answer any one question: $10 \times 1 = 10$

(a) (i) Apply Descarte's rule of sign to determine the nature of the roots of the equation $x^4 + 2x^2 + 3x - 1 = 0$.

(ii) Using principle of mathematical induction, prove that $5^n - 1$ is divisible by 4 for all positive integer n.

(iii) Find all eigen values of the matrix $A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{pmatrix}$. Hence find eigen vector corresponding to the smallest eigen value. 2+3+(2+3)=10

(b) (i) If α, β, γ are the roots of the equation $x^3 + qx + r = 0$ $(r \neq 0)$, find the equation whose roots are $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}, \frac{\beta}{\gamma} + \frac{\gamma}{\beta}, \frac{\gamma}{\alpha} + \frac{\alpha}{\gamma}$.

(ii) Find $mod\ z$ and $amp\ z$ when $z = (1 + i)^7$.

(iii) Determine the representative matrix of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (3x + z, -2x + y, -x + 2y + 4z)$, w.r.t. the standard basis of $\mathbb{R}^3$. 3+4+3=10

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