(iii) Determine the conditions for which the system

$$x + y + z = 1$$
$$x + 2y - z = b$$
$$5x + 7y + az = b^2$$

admits of (i) only one solution, (ii) no solution, (iii) infinitely many solution. 5

(b) (i) A linear mapping $T:\mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$T(x_1, x_2, x_3) = \left(2x_1 + x_2 - x_3,\ x^2 + 4x_3,\ x_1 - x_2 + 3x_3\right),$$

$(x_1, x_2, x_3) \in \mathbb{R}^3$.

Find the matrix to T relative to the ordered basis ((0,0,1), (1,0,0), (0,1,0)) of $\mathbb{R}^3$. 3

(ii) Check whether the relations p on $\mathbb{Z}$ defined by "$x\ p\ y$ *if f* $x|y$" is antisymmetric. 2

(iii) Find whether the following is bijective; if so, find their inverse.

Let $S = \{x \in \mathbb{R} : -1 < x < 1\}$ and $f:\mathbb{R} \to S$ defined by

$$f(x) = \frac{x}{1+|x|}.$$ 5

B.Sc. 1st Semester (Honours) Examination-2022-23

MATHEMATICS

Course ID : 12112 Course Code : SH/MTH/102/C-2

Course Title : Algebra (New)

Time : 2 Hours *Full Marks :* 40

The figures in the right hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meaning.

Unit–I

1. Answer **any *five*** questions : 2×5=10

(a) Find the product of all the values of $\left(\sqrt{3}+i\right)^{\frac{3}{5}}$.

(b) Find the dimension of the Subspace S of $\mathbb{R}^4$ defined by $S = \left\{(x, y, z, w) \in \mathbb{R}^4 : x + 2y - z = 0,\ \ 2x + y + w = 0\right\}$.

(c) Use Euclidean algorithm to find two integers u and v such that gcd (13,80)=13u + 80v.

(d) Determine a Linear mapping $T: R^3 \to R^3$ which maps the basis vectors {(0,1,1), (1,0,1) and (1,1,0)} to the vectors {(2,1,1), (1,2,1) and (1,1,2)} respectively.

(e) If α be an eigen value of an $n \times n$ idempotent matrix A, then show that α is either 0 or 1.

(f) examine if the relation p defined on $N \times N$ by "$(a,b)p(c,d)$ holds if and only if $ad = bc$" is reflexive, symmetric and transitive or not.

(g) Find the rank of the matrix $\begin{pmatrix} 0 & 2 & 4 & 2 & 2 \\ 4 & 4 & 4 & 8 & 0 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix}$

(h) Find the number of real positive and negative roots of the equation $x^3-7x+7=0$, using Strum's theorem.

Unit–II

2. Answer **any *four*** questions : 5×4=20

(a) (i) Show that the product of any m consecutive integers is divisible by m.

(ii) Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x^2 - 5$ and $g(x) = \frac{x}{x^2+1}$. Find fog and gof. Also show that $g^{-1}(g(2,3\})) \neq \{2,3\}$. 2+3

(b) Determine the condition for which the system of equations has (i) unique solution (ii) no solution and (iii) many solutions :

$x + 2y + z = 1,\ 2x + y + 3z = b,\ x + ay + 3z = b + 1$

(d) (i) Find the principal value of $(1 + i)^i$. 2

(ii) Find the general solution of $\cos z = 2$. 3

(e) Find the equation whose roots are the roots of $x^4 - 8x^2 + 8x + 6 = 0$, each diminished by 2. 5

(f) Show that eigen values of a real symmetric matrix are all real. 5

Unit–III

3. Answer **any *one*** question : 10×1=10

(a) (i) If $A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$, show that $A^2 - 10A + 16I_3 = 0$.

Hence obtain A^{-1}. 3

(ii) Find all real x for which the rank of the matrix is less than 4. 2

$$\begin{pmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{pmatrix}$$

(g) Apply Descartes' rule of signs to examine the nature of the roots of the equation $x^4 + 2x^2 + 3x - 1 = 0$.

(h) Use Cayley–Hamilton theorem to compute A^{-1} where $A = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$.

Unit–II

2. Answer **any *four*** questions : 5×4=20

(a) (i) Let $f : A \to B$ and $g : B \to C$ be both bijections. Prove that $g \circ f : A \to C$ is invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. 3

(ii) Suppose f and g are two functions from $\mathbb{R}$ into $\mathbb{R}$ such that $f \circ g = g \circ f$. Does it necessarily imply that $f = g$? Justify your answer. 2

(b) (i) Find gcd of 615 and 1080 and find integers s and t such that gcd (615,1080) = 615 s + 1080 t. 2

(ii) What is the remainder when 1! + 2! + 3! + ⋯ + 99! + 100! is divided by 18? 3

(c) A relation ρ is defined on $\mathbb{Z}$ by "$x \, \rho \, y$ if and only if $x^2 - y^2$ is divisible by 5" for $x, y, \in \mathbb{Z}$. Prove that ρ is an equivalence relation on $\mathbb{Z}$. Show that there are three distinct equivalence classes. 5

(c) (i) Find arg z where $z = 1 + i\tan\frac{3\pi}{5}$.

(ii) In n be a positive integer and $(1 + z)^n = p_0 + p_1 z + p_2 z^2 + \cdots + p_n z^n$. Prove that $p_0 - p_2 + p_4 - \cdots = 2^{n/2}\cos\frac{n\pi}{4}$ and $p_1 - p_3 - p_5 - \cdots = 2^{n/2}\sin\frac{n\pi}{4}$.

2+3

(d) (i) Let α, β, γ be the roots of the equation $x^3 + ax^2 + bx + c = 0$. Find the value of $\Sigma\alpha^2\beta^2$.

(ii) Solve the equation by Cardan's method : $x^3 - 27x - 54 = 0$. 2+3

(e) (i) Let A be a 3 × 3 matrix with eigen values are 1, –1,0. Find all eigen values of the matrix $I + A^{2022}$.

(ii) Find the algebraic and the geometric multiplicities of each eigen values of the matrix

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{pmatrix}$$

1+4

(f) (i) If x, y, z be three unequal positive numbers, then show that $\left(\frac{y+z}{2}\right)^x \left(\frac{z+x}{2}\right)^y \left(\frac{x+y}{2}\right)^z < x^x y^y z^z$.

(ii) Show that $2^{73} + 14^3 = 2 \bmod (11)$ 3+2

Unit–III

3. Answer **any *one*** question : 10×1=10

(a) (i) State Caley–Hamilton theorem for a matrix A.

Using this theorem find A^{50}, where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(ii) Show that every integer and its cube leave the same remainder when divided by 6.

(iii) Using the principle of Mathematical Induction, prove that $7^{2n} + 16n - 1$ is divisible by 64, for all $n \in N$. (1+3)+3+3

(b) (i) Apply Descarts rule of signs to find the nature of the roots of the equation $x^4 + 2x^4 + 3x - 1 = 0$.

(ii) Use principle of induction to prove that $2 \cdot 7^n + 3 \cdot 5^n - 5$ is divisible by 24 for all $n \in \mathbb{N}$.

(iii) Let the matrix of a linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ relative to the ordered bases ((0,1,1), (1,0,1), (1,1,0)) of $\mathbb{R}^3$ be $\begin{pmatrix} 0 & 3 & 0 \\ 2 & 3 & -2 \\ 2 & -1 & 2 \end{pmatrix}$. Find T. Find the matrix of T relative to the ordered basis (2,1,1), (1,2,1), (1,1,2) of $\mathbb{R}^3$. 2+3+5

Course Title : Algebra (Old)

Unit–I

1. Answer **any *five*** question : 2×5=10

(a) Find the remainder when 11^{40} is divided by 8.

(b) Let a, b, c be all positive real numbers. Then prove that

$$\frac{a^2 + b^2}{a + b} + \frac{b^2 + c^2}{b + c} + \frac{c^2 + a^2}{c + a} \geq a + b + c.$$

(c) Let $f : \mathbb{Z} \to E_+^0$ be defined by $f(x) = |x| + x \quad \forall x \in \mathbb{Z}$ *and* E_+^0 be the set of all nonnegative even integers. Let $g : E_+^0 \to \mathbb{Z}$ be defined by $g(x) = \frac{x}{2} \quad \forall x \in E_+^0$.

Check whether g is inverse of f.

(d) Check whether $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 4n - 5$ $\forall n \in \mathbb{Z}$ is bijective.

(e) If $a_1, a_2, \ldots, b_1, b_2, \ldots, b_n; c_1, c_2, \ldots, c_n$ are all positive real numbers then prove that $(a_1 b_1 c_1 + a_2 b_2 c_2 + \cdots + a_n b_n c_n)^2 < \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n^2\right)\left(c_1^2 + c_2^2 + \cdots + c_n^2\right)$.

(f) Solve the equation $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$.